ON THE CONNECTION BETWEEN THE RUNGE-KUTTA METHOD AND PICARD'S METHOD

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The Runge-Kutta method for the numerical solution of Cauchy's problem for a system of ordinary differential equations has an obvious iterative character. As demonstrated in this paper, this phenomenon arises from the connection between the Runge-Kutta method and Picard's iterative method. The estimation of error for the Runge-Kutta method is based on this connection.

1. Let Cauchy's problem be given for a normal system of ordinary differential equations of the first order

$$y_r'(x) = f_r(x; y_1, \ldots, y_n), \qquad y_r(x_0) = y_r^\circ \qquad (r = 1, \ldots, n) \tag{1.1}$$

Below we use the following notations:

integral form of the problem (1.1)

$$y_r(x) = y_r^\circ + \int_{x_0}^{x} f_r[t; y_1(t), \ldots, y_n(t)]\, dt \qquad (r = 1, \ldots, n) \tag{1.2}$$

approximate solution of Equation (1.2), obtained after s iterations by Picard's method

$$y_{r,s}(x) = y_r^\circ + \int_{x_0}^{x} f_r[t_s; y_{1,s-1}(t_s), \ldots, y_{n,s-1}(t_s)]\, dt_s \qquad (r = 1, \ldots, n) \tag{1.3}$$

approximate solution of the problem (1.1) by the Runge-Kutta method

$$Y_{r,s}(x) = y_r^\circ + h \sum_{i=1}^{s} \beta_{r,i} k_{r,i} \qquad (r = 1, \ldots, n) \tag{1.4}$$

Here

$$k_{r,i} = h f_r \Big\{x_0 + \alpha_{r,i} h;\ y_1^\circ + \sum_{j=1}^{i-1} \beta_{r,i,1,j} k_{1,j}, \ldots, y_n^\circ + \sum_{j=1}^{i-1} \beta_{r,i,n,j} k_{n,j}\Big\} \tag{1.5}$$

$$h = x - x_0, \quad \alpha_{r,1} = 0; \ \alpha_{r,i}, \quad \beta_{r,i,j,k} = \text{const} \quad (r = 1, \ldots, n), \quad (i = 1, \ldots, s),$$

Let us prove the theorem about the connection between the Runge-Kutta method and Picard's method.

Theorem 1. The solution of the problem (1.1) by means of the Runge-Kutta method (1.4) gives the same solution as that of the problem (1.2) by Picard's method (1.3), if the integrals in (1.3) are successively substituted by numerical quadratures.

P r o o f . Let us take $y_{r,0}(x) = y_r^\circ$ for the zero approximation in (1.3) and decompose the interval of the integration $[x_0, x]$ for every Equation (1.3) by the same method into s intervals by the points

$$x_0 + \alpha_{r,i}h \qquad (i = 1, 2, \ldots, s+1)$$

taking $\alpha_{r,1} = 0$ and $\alpha_{r,s+1} = 1$. Let us use the approximation formula

$$\int\limits_{x_0+\alpha_{r,i}h}^{x_0+\alpha_{r,i+1}h} F(x)\,dx \approx \beta h F(x_0 + \alpha_{r,i}h) \tag{1.6}$$

According to the above presentation we have

$$y_{l,k}(x_0 + \alpha_{r,i}h) = y_l^\circ + \int\limits_{x_0}^{x_0+\alpha_{r,i}h} f_l\,[t_k;\, y_{1,k-1}(t_k), \ldots, y_{n,k-1}(t_k)]\,dt_k = \tag{1.7}$$

$$= y_l^\circ + \sum_{j=1}^{i-1} \int\limits_{x_0+\alpha_{r,j}h}^{x_0-\alpha_{r,j+1}h} f_l\,[t_k;\, y_{1,k-1}(t_k), \ldots, y_{n,k-1}(t_k)]\,dt_k \approx Y_{l,k}(x_0 + \alpha_{r,i}h) =$$

$$= y_l^\circ + \sum_{j=1}^{i-1} \beta_{r,i,l,j} h f_l\,[x_0 + \alpha_{r,j}h];\quad Y_{1,k-1}(x_0 + \alpha_{r,j}h), \ldots, Y_{n,k-1}(x_0 + \alpha_{r,}\,h)]$$

Here $Y_{l,k}(x_0 + \alpha_{r,i}h)$ is the approximate value of

$$y_{l,k}(x_0 + \alpha_{r,i}h),\ r = 1, 2, \ldots, n;$$

$$l = 1, 2, \ldots, n;\quad k = 1, 2, \ldots, s;\quad i = 1, 2, \ldots, s+1;\ Y_{r,0} = y_{r,0} = y_r^\circ,$$

with the understanding, that in the case $j = 1$ the sum is empty.

After the remark

$$Y_{l,k}(x_0 + a_1 h) = Y_{l,k}(x_0) = y_{l,0} \qquad (l = 1, 2, \ldots, n;\ k = 0, 1, \ldots, s) \tag{1.8}$$

we prove the validity of Equation

$$Y_{l,s-1}(x_0 + \alpha_{r,i}h) = Y_{l,s-2}(x_0 + \alpha_{r,i}h) \qquad \begin{pmatrix} l = 1, 2, \ldots, n \\ r = 1, 2, \ldots, n \end{pmatrix} \tag{1.9}$$

for $i = 1, 2, \ldots, s-1$.

For the sake of brevity we shall omit inside f the argument of $Y_{i,j}$ and provide Y with the index σ , assuming that simultaneously it substitutes $s-1$ and $s-2$. Thus the expression $Y_{l,\sigma}$ has to be read once as $Y_{l,s-1}$ and secondly as $Y_{l,s-2}$.

Now let us write (1.7) fir $k = s-1$ and $k = s-2$ consecutively in steps.

First step

$$Y_{l,\sigma}(x_0 + \alpha_{r,i}h) = y_l^\circ + \sum_{j=1}^{i-1} h\beta_{r,i,l,j} f_l\,[x_0 + \alpha_{r,j}h;\, Y_{1,\sigma-1}, \ldots, Y_{n,\sigma-1}]$$

Second step

$$Y_{l,\sigma}(x_0 + \alpha_{r,i}h) = y_l^\circ + \sum_{j=1}^{i-1} h\beta_{r,i,l,j} f_l \Big\{x_0 + \alpha_{r,j}h;\, y_1^\circ +$$

$$+ \sum_{k=1}^{j-1} \beta_{r,j,1,k} h f_1 (x_0 + \alpha_{r,k}h;\, Y_{1,\sigma-2}, \ldots, Y_{n,\sigma-2}), \ldots, y_n^\circ +$$

$$+ \sum_{k=1}^{j-1} \beta_{r,j,n,k} h f_n (x_0 + \alpha_{r,k}h;\, Y_{1,\sigma-2}, \ldots, Y_{n,\sigma-2})\Big\} \tag{1.10}$$

Continuing this writing, we realize that with every step the set of members of the innermost sums decreases by one, and also the second index of $Y_{i,j}$ lowers itself by one. Furthermore, $Y_{i,s-1}$ is different from $Y_{i,s-2}$ only with respect to Y in the innermost sums.

It follows from the above that the innermost sums of the step $t = 1$ have the form

$$\sum_{l=1}^{1} \beta_{r,2,k,l} h f_k (x_0 + \alpha_{r,l} h; Y_{1,\sigma-i+1}, \ldots, Y_{n,\sigma-j+1}) =$$

$$= \beta_{r,2,k,1} h f_k [x_0 + \alpha_{r,1} h; Y_{1,\sigma-i+1}(x_0 + \alpha_{r,1} h), \ldots, Y_{n,\sigma-i+1}(x_0 + \alpha_{r,1} h)]$$

However, according to (1.8)

$$Y_{k,s-i}(x_0 + \alpha_{r,1} h) = Y_{k,s-i-1}(x_0 + \alpha_{r,1} h) \qquad \begin{pmatrix} k = 1, 2, \ldots, n \\ i = 1, 2, \ldots, s-1 \end{pmatrix}$$

Thus the Theorem (1.9) is proved. Now we write (1.3) in the form

$$y_{r,s}(x) = y_{r,s}(x_0 + \alpha_{s+1} h) = \qquad (r = 1, 2, \ldots, n) \tag{1.11}$$

$$= y^\circ + \sum_{i=1}^{s} \int_{x_0+\alpha_{r,i} h}^{x_0+\alpha_{r,i+1} h} f_r [t_s; y_{1,s-1}(t_s), \ldots, y_{n,s-1}(t_s)]\, dt_s$$

After substituting the integral in (1.11) by numerical quadratures along with (1.6) and (1.7) we obtain

$$y_{r,s}(x) \approx Y_{r;s}(x) = y_r^\circ + \sum_{i=1}^{s} h\beta_{r,i} f_r [x_0 + \alpha_{r,i} h; \tag{1.12}$$

$$Y_{1,s-1}(x_0 + \alpha_{r,i} h), \ldots, Y_{n,s-1}(x_0 + \alpha_{r,i} h)] = y_r^\circ + \sum_{i=1}^{s} \beta_{r,i} k_{r,i} \quad (r = 1, 2, \ldots, n)$$

It is clear, that

$$k_{r,i} = h f_r [x_0 + \alpha_{r,i} h; Y_{1,s-1}(x_0 + \alpha_{r,i} h), \ldots, Y_{n,s-1}(x_0 + \alpha_{r,i} h)]$$

$$(r = 1, 2, \ldots, n;\ i = 1, 2, \ldots, s) \tag{1.13}$$

or

$$k_{r,i} = h f_r \Big\{ x_0 + \alpha_{r,i} h; y_1^\circ +$$

$$\sum_{j=1}^{i-1} \beta_{r,i,1,j} h f_1 [x_0 + \alpha_{r,j} h; Y_{1,s-2}(x_0 + \alpha_{r,j} h), \ldots, Y_{n,s-2}(x_0 + \alpha_{r,j} h)], \ldots, y_n^\circ +$$

$$+ \sum_{j=1}^{i-1} \beta_{r,i,n,j} h f_n [x_0 + \alpha_{r,j} h; Y_{1,s-2}(x_0 + \alpha_{r,j} h), \ldots Y_{n,s-2}(x_0 + \alpha_{r,j} h)] \Big\}$$

Since here $j \leqslant i - 1 \leqslant s - 1$, we may use the relations (1.9) and write the expression obtained for $k_{r,i}$ in the form

$$k_{r,i} = h f_r \{ x_0 + \alpha_{r,i} h; y_1^\circ + \tag{1.14}$$

$$+ h \sum_{j=1}^{i-1} \beta_{r,i,1,j} f_1 [x_0 + \alpha_{r,j} h; Y_{1,s-1}(x_0 + \alpha_{r,j} h), \ldots Y_{n,s-1}(x_0 + \alpha_{r,j} h)], \ldots, y_n^\circ +$$

$$+ h \sum_{j=1}^{i-1} \beta_{r,i,n,j} f_n [x_0 + \alpha_{r,j} h; \quad Y_{1,s-1}(x_0 + \alpha_{r,j} h), \ldots, Y_{n,s-1}(x_0 + \alpha_{r,j} h)] \}$$

or, substituting (1.13) in (1.14), we obtain finally

$$k_{r,i} = h f_r \Big(x_0 + \alpha_{r,i} h; y_1^\circ + \sum_{j=1}^{i-1} \beta_{r,i,1,j} k_{1,j}, \ldots, y_n^\circ + \sum_{j=1}^{i-1} \beta_{r,i,n,j} k_{n,j} \Big)$$

$$(r = 1, \ldots, n; \quad i = 1, \ldots, s) \tag{1.15}$$

q.e.d.

2. Let us pass over to the estimation of error of the approximate solution of problem (1.2), using Picard's iterative method of substituting integrals by numerical quadratures. As in Section 1 we shall do s iterations, take the vector function $v(x)$ for zero approximation and substitute the integrals by some numerical quadratures.

The estimation of error by the Runge-Kutta method mentioned in Section 1 will be a particular case of the estimation here obtained.

The basic idea of the estimation of error will be obtained from the proof of Picard's theorem on the existence and uniqueness of the solution of Cauchy's problem for a normal system of ordinary differential equations (compare e.g. [1], pp. 9-16). However, in the above mentioned proof there are introduced from the very beginning rough estimations, in consequence of which in some cases the error will be overestimated. In our considerations here the overestimation is due to the method applied. In practice, for the purpose of simplifying the estimation of error, the estimation in every concrete case needs only be considered with reference to the required accuracy of the approximate solution. For instance, in the case of the solution of a differential equation of order n, it is important to know and to keep within certain limits only the error of the unknown function, while the derivatives of the solution are not necessary. According to this requirement it will be necessary to simplify the estimation of error.

The rounding-off errors and the calculations of the right-hand sides of the system (1.1) will not be considered, since the analysis of their influence on the error of the numerical solution has been made in detail in [3].

For the sake of brevity let us use vector notation. Let $|v|$ denote an n-dimensional vector, the coordinates of which are absolute values of the coordinates of the vector v. If all coordinates of the vector d are larger than the corresponding coordinates of the vector b of the same dimension as d, we will write $d > b$. In vector notation (1.2) and (1.3) become

$$y(x) = y^\circ + \int_{x_0}^{x} f[t, y(t)]\,dt, \qquad y_s(t) = y^\circ + \int_{x_0}^{x} f[t, y_{s-1}(t)]\,dt \tag{2.1}$$

Now let us define the domain D by the inequalities

$$x_0 \leqslant x \leqslant x_0 + a, \qquad |y - y^\circ| \leqslant b \tag{2.2}$$

We suppose that in the domain D the vector function $f(x,y)$ is continuous and fulfills the Lipschitz condition of the first order in the variable y. Consequently, there exist a vector c and the matrix P with nonnegative components and with elements such, that in D

$$|f(x, y)| \leqslant c, \qquad |f(x, y) - f(x, z)| \leqslant P\,|_1 y - z| \tag{2.3}$$

On the interval $x_0 \leqslant x \leqslant x_0 + \delta$, where δ is chosen such, that the inequalities

$$0 < \delta \leqslant a, \qquad \delta c \leqslant b \tag{2.4}$$

are simultaneously satisfied; the error $\varepsilon_1(x) = |y(x) - y_s(x)|$ can be obtained in the following way ([2], p.110); according to the definition of Picard's iteration we have

$$y_i(x) = y^\circ + \int_{x_0}^{x} f[t, y_{i-1}(t)]\,dt \qquad (i = 1, 2, \ldots), \qquad y_0(x) = v(x) \in D \tag{2.5}$$

From this we may get with the help of the second inequality (2.4)

$$|y_i(x) - y_{i-1}(x)| \leqslant \frac{(x - x_0)^{i-1}}{(i-1)!} P^{i-1} \max |y_1(x) - v(x)| \quad (i = 1, 2, \ldots) \tag{2.6}$$

and for $k > s$

$$|y_k(x) - y_s(x)| \leqslant |y_k(x) - y_{k-1}(x)| + |y_{k-1}(x) - y_{k-2}(x)| + \ldots +$$
$$+ |y_{s+1}(x) - y_s(x)| \leqslant \sum_{i=s}^{k-1} \frac{(x - x_0)^i}{i!} P^i \max |y_1(x) - v(x)| \tag{2.7}$$

Passing to the limit in the last inequality for $k \to \infty$, we obtain

$$\varepsilon_1(x) \leqslant \left[e^{P(x-x_0)} - \sum_{i=0}^{s-1}\frac{(x-x_0)^i P^i}{i!}\right] \max|y_1(x) - v(x)| \qquad (2.8)$$

Substituting consecutively the integrals in (2.2) by the numerical quadrature formulas, we get the solution $Y(x)$

The error $\varepsilon_2(x) = |y_s(x) - Y(x)|$ of the numerical quadrature formulas will be assumed as known. Then the error $\varepsilon(x) = |y(x) - Y(x)|$ will have the estimation

$$\varepsilon(x) \leqslant \varepsilon_1(x) + \varepsilon_2(x) \qquad (2.9)$$

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